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Formulas on Queues in Burst Processes—I

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Queues arising in buffers due to either random interruptions of the channel or variable source rates are analyzed in the framework of a single switched system. Examples of systems to which the results of the paper may be applied are: multiplexing of speech with data in telephone channels and, in certain instances, buffering of data generated by the coding of moving images in the Picturephone® *system. The switched system consists of a uniform source, buffer, switch and channel. The source feeds data to the buffer at a uniform rate. The buffer's access to the channel is controlled by the switch; if the switch is closed, the buffer empties to the extent of the channel's transmission rate. The on-off pattern of the switch is indicated by a* $0 - 1$ *burst process* $\{E_j\}$, $j = 0, 1, 2, \cdots$; *if* $E_j = 0$, *the switch is closed for the duration* $[j, j+1)$. *The burst phenomenon is introduced to account for two different processes responsible for the event* $E_j = 0$. *There are relatively long periods during which* $E_j = 0$ *uniformly, and the activity separated by such periods is defined to be a burst. During a burst,* $E_j = 0$ *only infrequently. The duration of a burst is an independently distributed random variable with a geometric or weighted sum of geometric distributions. The inter-burst periods are assumed to be sufficiently long for the buffer to empty at some point during these periods of inactivity. During a burst* $\{E_j\}$ *is a Bernoulli sequence of independent random variables.*

Exact expressions for a variety of performance functionals related to the system described above are obtained, together with qualitative results. Recursive formulas are obtained for the following: (i) steady-state distribution of buffer content for a finite buffer of size N; *(ii) mean time for first passage across a level* N; *(iii) the probability of overflow, for a given level* N, *during a burst; (iv) mean time for first passage across a level* N *during a burst. The recursion in each case is with respect to* N. *The asymptotic behavior of the main recursions is determined.*

* The sequence of names was determined by coin tossing.

I. INTRODUCTION

A convenient framework for an unified analysis of a variety of digital communication systems involving huffering–some are discussed later–is provided by the system in Fig. 1. The source emits data uniformly at the rate of one symhol per unit time. The transmission rate of the channel is $(k + 1)$ symbols per unit time where k is some positive integer. Tbe buffer has access to tbe channel only when the switch is closed. The switch is controlled by a hurst process $\{E_j\}$, $j = 0, 1, 2, \cdots$. E_j, for every j, is either 0 or 1. If $E_j = 0$, the switch is closed for the duration $[j, j + 1)$; otherwise the switch is open. The burst process is introduced to account for cases where two hasically different types of phenomena are responsihle for the event $E_j = 0$. There are relatively long periods during wbich $E_j = 0$ uniformly; the activity separated hy such periods is defined to be a burst. On tbe other hand, during a burst, $E_j = 0$ only infrequently. The duration or length of a burst is a random variable. It is assumed that the burst length is independently distributed witb a geometric or a weighted sum of geometric distributions. The interburst periods are assumed to he sufficiently long for the huffer to empty during these periods. The statistical assumption made in the paper ahout the controlling sequence $\{E_j\}$ within a hurst is that it is a Bernoulli sequence of independent random variahles and $\Pr\{E_j = 1\} = \pi$ where $0 < \pi < 1$. In a companion paper, the case where $\{E_j\}$ is first-order Markov will be considered.

Important aspects of various digital communication systems are suhsumed within the framework of the system descrihed above. Diverse schemes for multiplexing data with speech on telephone channels[1,2] are representative of one class of such systems. A summary of the main features of the system which has heen described in some detail in Ref. 1 follows. The central idea is to utilize the telephone channel during the periods of silence in speech which amount to as much as half of the total conversation period to transmit digital data. The speaker needs to have priority for tbe use of the cbannel since otherwise the quality of speech is impaired. $E_j = 0(1)$ corresponds to the decision that silence (speech) exists during the interval $[j, j + 1)$

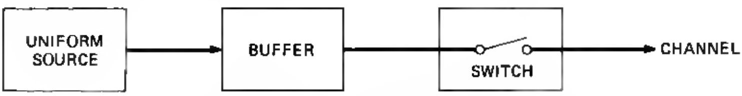

Fig. 1—Switched communication system.

so that only after it is decided that the speaker is silent does the buffer have access to the channel. An excellent example of the burst phenomenon may be found in speech monologues. Due to the presence of phrases in speech, two types of silences, interphrase and intraphrase silences, exist;[3,4] the former type consists of silences no less than 250 ms long and this is substantially longer than the mean duration of (uninterrupted) intraphrase silence.

There exists another class of digital communication systems composed of systems with only one source which transmits at a nonuniform rate. Most of the time the source rate is less than, say, r_0 bits per unit time and r_0 is less than the channel rate r. Occasionally, for short periods of time, the source rate spurts to a level r_1 which exceeds r. During such periods buffering becomes necessary. These occasional bursts of overloading of the channel are indicated by the $\{E_j\}$ process. The relation to the switched communication system of Fig. 1 is clear if $(r_1 - r)$ is normalized to unity, and $(r - r_0)$ corresponds to k.

An example of such a system for wbich the analysis of tbis paper is relevant arises in buffering of data generated by the coding of moving images in the *Picturephone*® system.[5] In this case, of course, r_0 and r_1 should be interpreted as average rates[6] in the two regimes, or, when the worst case is of interest, as the extreme rates. The results of this paper appear to be relevant[7] for variable rate in-frame coding, since during bursts of bigh detail, the correlation of the data rates for successive picture elements is not high. For frame-to-frame coding the first-order Markov model, to be treated in a companion paper, is of interest.

Exact expressions for diverse performance functionals related to the system in Fig. 1 are obtained, together with qualitative results. As a whole they provide a rather comprehensive set of criteria for the design of the important parameters of the system, such as the buffer size and the transmission rate of the channel. A summary of the main contributions follows.*

(*i*) A recursive formula is obtained for the steady-state distribution of buffer content for finite buffers. The recursion is with respect to N wbere N is tbe size of the buffer.

(*ii*) It is proved that F_N, the mean time for first passage through a level N, is given by

$$F_N = \frac{1}{\pi}F_{N-1} - \frac{1-\pi}{\pi}F_{N-k-1} + \frac{1}{\pi}\cdot$$

* N is used to denote both the buffer size [as in (*i*), (*iii*) and (*vi*)] and a level [as in (*ii*) and (*iv*)]. In what follows, the definition of N should be clear from the context.

(*iii*) g_N, the probability of overflow of a buffer of size N during a burst, the duration of which is distributed geometrically with a parameter ρ is given by

$$g_N = \left(\frac{1}{\rho\pi}\cdot\frac{1}{g_{N-1}} - \frac{1-\pi}{\pi}\cdot\frac{1}{g_{N-k-1}}\right)^{-1}.$$

(*iv*) A closed expression and a recursive formula are obtained for the mean time for first passage through a level N *during a burst;* the recursion is with respect to N.

(*v*) The asymptotic behavior of all formulas in items (*i*) through (*iii*), as N becomes large, is given. The derivation is dependent on the following: of the roots of the polynomials associated with the recursions, either one or two roots, depending on which recursion is being considered, lie outside the unit circle.

(*vi*) It is proved that under certain conditions on the initial probability distribution of the contents of the buffer, the probability of a buffer being full is a monotonic, nondecreasing sequence with respect to time; if the buffer is initially empty, the above-mentioned conditions are satisfied. One of the main implications of the result is that the steady-state probability of the buffer being full is an upper bound on the probabilities of the buffer being full at any instant. Furthermore, a particularly simple recursion is obtained for P_N, the steady-state probability of a buffer of size N being full:

$$P_N = \left(\frac{1}{\pi}\cdot\frac{1}{P_{N-1}} - \frac{1-\pi}{\pi}\cdot\frac{1}{P_{N-k-1}}\right)^{-1}.$$

(Observe that $1/P_N$ is also the mean time for recurrence of the state corresponding to a full buffer.)

The closed expressions obtained are for all k and N, and the recursions hold for all $N \geqq 2k+1$. Wherever applicable, the buffer is assumed to be initially empty. An important feature of the given formulas is that they are also given in the form of recursions. The advantages of recursive formulas over the alternate versions need to be emphasized. For a given N, typically, a closed expression for a recursion involves inverting a matrix of order N. For large N, the effort is substantial. If, in addition, it is borne in mind that a designer is interested in functionals associated with a range of possible buffer sizes, the advantages of recursive formulas of the form given in this paper become overwhelming. This is only to be expected since the recursions are obtained by taking into full account the structure of the matrices involved.

II. EQUATIONS OF PROCESSES

Let B_j be the number of symbols in the buffer at the jth instant. For a finite buffer of size N,

$$\begin{aligned} B_{j+1} &= \mathrm{Max}\{B_j - k, 0\} \quad \text{if} \quad E_j = 0 \\ &= \mathrm{Min}\{B_j + 1, N\} \quad \text{if} \quad E_j = 1. \end{aligned}$$

Since B_j depends only on B_{j-1} and E_{j-1}, the state of the Markov chain of interest at the jth instant, S_j, is determined by the value of B_j where $B_j \in \{0, 1, 2, \cdots, N\}$. Let $P_m(n)$ denote the probability of the state $S_m = n$. Then

$$P_m(0) = (1 - \pi) \sum_{j=0}^{k} P_{m-1}(j) \tag{1}$$

$$P_m(i) = \pi P_{m-1}(i - 1) + (1 - \pi) P_{m-1}(i + k) \qquad i = 1, 2, \cdots, N - k \tag{2}$$

$$P_m(i) = \pi P_{m-1}(i - 1) \qquad i = N - k + 1, N - k + 2, \cdots, N - 1 \tag{3}$$

$$P_m(N) = \pi[P_{m-1}(N - 1) + P_{m-1}(N)]. \tag{4}$$

It is well known from the theory of Markov chains [8] that the limiting distribution of the states $P(i)$ is obtained from (1) through (4) by equating $P_m(i)$ and $P_{m-1}(i)$ to $P(i)$.

2.1 *Equations for Some New Probabilities*

Central to most of what follows are the probabilities $Q_m(i)$, where

$$Q_m(i) = \Pr\{(S_m = i) \cap (B_j \leqq N, j \leqq m)\}$$

and the buffer size exceeds N. For convenience, let X_m denote the event $S_j \in \{0, 1, 2, \cdots, N\}$ for all j, $0 \leqq j \leqq m$, so that

$$Q_m(i) = \Pr\{(S_m = i) \cap X_m\}. \tag{5}$$

The equation governing the transitions of $\{Q_i\}$ is derived. It is shown that there exists a matrix A which relates $\{Q_i\}$ to $\{Q_{i-1}\}$, i.e.,

$$Q_i(j) = \sum_{l=0}^{N} A_{jl} Q_{i-1}(l) \tag{6}$$

or, in matrix notation, $Q_i = AQ_{i-1}$.

In (5) $i \in \{0, 1, \cdots, N\}$, so that

$$
\begin{aligned}
Q_m(i) &= \Pr\{(S_m = i) \cap X_{m-1}\} \\
&= \sum_{j=0}^{N} \Pr\{(S_m = i) \cap X_{m-1} \cap (S_{m-1} = j)\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Q_m(i) &= \sum_{j=0}^{N} \Pr\{(S_m = i) \mid (S_{m-1} = j) \cap X_{m-1}\} \Pr\{(S_{m-1} = j) \cap X_{m-1}\} \\
&= \sum_{j=0}^{N} \Pr\{(S_m = i) \mid (S_{m-1} = j) \cap X_{m-1}\} Q_{m-1}(j) \\
&= \begin{cases}
(1-\pi) \sum_{j=0}^{k} Q_{m-1}(j) \quad \text{if} \quad i = 0 & \text{(7a)} \\
\pi Q_{m-1}(i-1) + (1-\pi) Q_{m-1}(i+k) & \\
\qquad\qquad \text{if} \quad i = 1, 2, \cdots, N-k & \text{(7b)} \\
\pi Q_{m-1}(i-1) & \\
\qquad\qquad \text{if} \quad i = N-k+1, N-k+2, \cdots, N. & \text{(7c)}
\end{cases}
\end{aligned}
$$

(7) defines the $(N+1)$ by $(N+1)$ matrix A. Sometimes when the need arises, the $(N+1)$ by $(N+1)$ matrix A associated with a given N will be specified by $A(N)$.

$$
A = \begin{array}{cc}
\begin{array}{ccccccc} 1 & 2 & \cdots & k+1 & k+2 & \cdots & N+1 \end{array} & \\
\left[\begin{array}{ccccccc}
(1-\pi) & (1-\pi) & \cdots & (1-\pi) & & & \\
\pi & 0 & & & (1-\pi) & & \\
& \ddots & & & & \ddots & \\
& & \ddots & & & & (1-\pi) \\
& & & \ddots & & & \\
& & & & \pi & & \\
& & & & \cdot\,\pi & 0 & 0 \\
& & & & & \pi & 0
\end{array}\right] &
\begin{array}{c} 1 \\ 2 \\ \\ N-k+1 \\ \\ \\ N \\ N+1 \end{array}
\end{array}
\qquad (8)^*
$$

It will be observed that the only difference between (7) and the transition equations (1) through (4) for a finite buffer of size N, is that eq. (4) is modified since a transition from state N to state N is not possible in the present context. For the same reason, the matrix A is not a Markov matrix since the sum of the elements of the last, i.e., $(N+1)$th column is $(1-\pi)$, the remaining columns sum to unity as is the case for all columns of Markov matrices.

* The dots indicate continuation of the values of the adjoining elements; remaining elements are assumed to be 0.

If the transition matrix of the basic Markov process, i.e., the matrix defined by eqs. (1) through (4), is irreducible, then $(I - \rho A)$ is nonsingular for $|\rho| \leqq 1$. The proof follows from a well-known result in matrix theory[9] which in this case states tbat if all the columns of $(I - \rho A)$ are weakly column-sum dominant and at least one column of $(I - \rho A)$ is strongly column-sum dominant, then the matrix is nonsingular.

III. STEADY-STATE PROBABILITIES FOR FINITE BUFFERS

In this section, a formula is given for recursively generating tbe steady-state probabilities $P(i)$ where the recursion is, with respect to N, tbe size of the buffer. To distinguish the steady-state probabilities for different buffer sizes, the symbol $P^N(i)$ is introduced to denote $P(i)$ for a buffer of size N.

If $N \geqq k + 1$, as is almost always the case, an equation of the type given in (2), namely,

$$P^N(i - 1) - \frac{1}{\pi}P^N(i) + \frac{1 - \pi}{\pi}P^N(i + k) = 0 \tag{9}$$

occurs at least once and since $N \gg k$ usually, the main body of equations defining the steady-state probabilities is of that form. It is proved in Ref. 1 what may reasonably be expected, namely, every solution of the homogenous set of equations that define tbe steady-state probabilities is of the form

$$P^N(j) = \sum_{i=1}^{k+1} b_i \mu_i^{N-j} \qquad j = 0, 1, \cdots, N \tag{10}$$

where μ_i are the simple roots of the polynomial

$$\mu^{k+1} - \frac{1}{\pi}\mu^k + \frac{1 - \pi}{\pi}. \tag{11}$$

If the polynomial has multiple roots the obvious modifications must be made. [Note: Since $0 < \pi < 1$, the polynomial in (11) has distinct roots whenever $\pi \neq k/(k + 1)$; when $\pi = k/(k + 1)$, the only repeated root is 1.]

The complete recursive formula for $P^N(j)$ is obtained in two parts. First, a recursive formula for a set of solutions $q_N(j)$ to the steady-state equations is obtained and, second, a recursive formula for the

normalizing constant Σ_N is obtained. Finally,

$$P^N(j) = \frac{1}{\Sigma_N} q_N(j) \quad j = 0, 1, \cdots, N. \tag{12}$$

3.1 *Recursions for* $\{q_N(j)\}$

Let

$$q_N(N) = 1 \tag{13}$$

and suppose $\{q_N(j)\}$ satisfies the steady-state equations of a finite buffer of size N. Hence, $q^N(j)$ has the form given in (10).* For fixed N and $i = 1, 2, \cdots, k+1$, let

$$d_i \triangleq \sum_{j=1}^{k+1} a_j \mu_j^{i-1}.$$

The transformation $\{a_j\} \to \{d_i\}$ is invertible since the Vandermonde matrix is nonsingular. Now,

$$\begin{aligned} d_i &= \sum_{j=1}^{k+1} a_j \mu_j^{N-(N-i+1)} \\ &= q_N(N-i+1) \qquad i = 1, 2, \cdots, k+1. \end{aligned} \tag{14}$$

Also, from the steady-state equations themselves,

$$d_1 = q_N(N) = 1 \tag{15}$$

$$d_i = q_N(N-i+1) = \frac{1-\pi}{\pi^{i-1}} \qquad i = 2, 3, \cdots, k+1. \tag{16}$$

Hence, significantly, $\{d_i\}$ is independent of N from which it follows that $\{a_i\}$ is also independent of N.

$$\begin{aligned} q_{N+k+1}(j) &= \sum_{i=1}^{k+1} a_i \mu_i^{N+k+1-j} \\ &= \sum_{i=1}^{k+1} a_i \left\{ \frac{1}{\pi} \mu_i^{(N+k)-j} - \frac{1-\pi}{\pi} \mu_i^{N-j} \right\} \qquad \text{from (11)} \\ &= \frac{1}{\pi} q_{N+k}(j) - \frac{1-\pi}{\pi} q_N(j) \qquad j = 0, 1, \cdots, N. \end{aligned} \tag{17}$$

The formula for $\{q_{N+k+1}(j)\}$ is complete if (15) and (16) are appended,

* To distinguish between $\{P^N(j)\}$ and $\{q_N(j)\}$, denote the coefficients in the form for the latter by $\{a_i\}$, i.e., $b_i = (1/\Sigma_N)a_i$.

i.e.,

$$q_{N+k+1}(N+i) = \frac{1-\pi}{\pi^{k-i+1}} \qquad i = 1, 2, \cdots, k \tag{16}$$

$$q_{N+k+1}(N+k+1) = 1. \tag{15}$$

3.2 *Recursion for the Normalizing Constant*

Let

$$\Sigma_N \triangleq \sum_{j=0}^{N} q_N(j). \tag{18}$$

Now

$$\begin{aligned} \sum_{j=N+1}^{N+k+1} q_{N+k+1}(j) &= 1 + (1-\pi) \sum_{i=1}^{k} \left(\frac{1}{\pi}\right)^i \\ &= \frac{1}{\pi^k}. \end{aligned} \tag{19}$$

Summing both sides of (17),

$$\Sigma_{N+k+1} - \frac{1}{\pi^k} = \frac{1}{\pi}\left[\Sigma_{N+k} - \frac{1}{\pi^{k+1}}\right] - \frac{1-\pi}{\pi}\Sigma_N$$

i.e.,

$$\Sigma_{N+k+1} = \frac{1}{\pi}\Sigma_{N+k} - \frac{1-\pi}{\pi}\Sigma_N. \tag{20}$$

(20) is the recursion for the normalizing constant. The derivation of the recursive formula for $\{P^N(j)\}$ is now complete.

Observe that in the course of tbe above analysis, a simple recursive formula for the rather important steady-state probability of the buffer being full, i.e., $P^N(N)$, has been obtained.

$$P^N(N) = \frac{q_N(N)}{\Sigma_N} = \frac{1}{\Sigma_N} \tag{21}$$

and Σ_N, of course, is obtained from (20).

IV. MEAN FIRST PASSAGE TIME

Suppose N is a fixed positive integer and the buffer capacity is greater than N. A functional that provides substantial insight into the problem of designing a buffer for whicb tbe probability of overflow is small is F_N, the mean time required for tbe buffer contents to first exceed N. It is particularly useful in the context of burst processes wbere only incomplete data are available concerning tbe burst length

distribution–provided that the length of bursts is bounded, a simple comparison of the bound with F_N provides an useful guide. In this section a recursive formula for F_N, the recursion being with respect to N, is obtained. To correspond with the practical situation, the buffer is initially assumed to be empty; the same recursive formula holds for the other interesting initial condition, namely, the buffer initially contains an unit symbol.

X_m is the event that $S_j \in \{1, 2, \cdots, N\}$ for all j, $0 \leqq j \leqq m$.

$$O_i \triangleq \Pr\{\text{overflow occurs for the first time at } i\} \tag{22}$$

$$\begin{aligned} &= \Pr\{(E_{i-1} = 1) \cap (S_{i-1} = N) \cap X_{i-1}\} \\ &= \pi \Pr\{(S_{i-1} = N \cap X_{i-1})\}, \text{ from the independence of } \{E_i\} \\ &= \pi Q_{i-1}(N) \end{aligned} \tag{23}$$

where $\{Q_i\}$ is as defined in eq. (5). It has been shown in Section 2.1 that

$$Q_i = AQ_{i-1}. \tag{6}$$

Hence,

$$\begin{aligned} O_i &= \pi Q_{i-1}(N) \\ &= \pi(0, \cdots, 0, 1)Q_{i-1} \\ &= \pi(0, \cdots, 0, 1)A^{i-1}Q_0 \\ &= \pi e_r^t A^{i-1} Q_0 \end{aligned} \tag{24}$$

where e_i denotes the vector* with a single element equal to unity at the ith location and all remaining elements 0; $r = N + 1$. Finally,

$$F_N = \text{Mean time for first passage through level } N$$

$$= \sum_{i=1}^{\infty} iO_i \tag{25}$$

$$\begin{aligned} &= \pi \sum_{i=1}^{\infty} ie_r^t A^{i-1} Q_0 \\ &= \pi e_r^t \left(\sum_{i=1}^{\infty} iA^{i-1} \right) Q_0 \\ &= \pi e_r^t (I - A)^{-1}(I - A)^{-1} Q_0. \end{aligned} \tag{26}$$

Let

$$x^t \triangleq e_r^t (I - A)^{-1}$$

* The superscript t denotes the transpose of a vector.

so that

$$x^t(I - A) = e_r^t. \tag{27}$$

But

$$x^t = \frac{1}{\pi}(1, 1, \cdots, 1) \tag{28}$$

is a solution of (27) since the elements of the last, i.e., $(N + 1)$th column of A sum to $(1 - \pi)$ and the remaining columns sum to 1. Moreover, (28) is the unique solution of (27) since $(I - A)$ is non-singular. Hence,

$$\begin{aligned} F_N &= (1, 1 \cdots, 1)(I - A)^{-1}Q_0 \\ &= 1^t(I - A)^{-1}Q_0 \end{aligned} \tag{29}$$

where 1 denotes the vector with all elements equal to unity. In the following section, the above formula with $Q_0 = e_1$ is analyzed further to yield a recursive formula.

4.1 *Recursive Formula for F_N*

Henceforth, it is necessary to be specific about the dimensions of A–the matrix A associated with a given N is denoted by $A(N)$. The buffer is assumed to be initially empty, i.e., $S_0 = 0$ or, equivalently, $Q_0 = e_1$.

Since* $|I - A(N)|[I - A(N)]^{-1}e_1$ is the vector of (signed) cofactors of the 1st row of $[I - A(N)]$

$$|I - A(N)|1^t[I - A(N)]^{-1}e_1 = |D(N)| \tag{30}$$

where $D(N)$ is the $(N + 1)$ by $(N+1)$ matrix obtained from $A(N)$ by replacing all elements of the first row of $A(N)$ by unity. Then, from (29),

$$F_N = \frac{|D(N)|}{|I - A(N)|}. \tag{31}$$

Adding rows 2, 3, $\cdots$, $(N + 1)$ of $[I - A(N)]$ to the first row, it can be verified that

$$|I - A(N)| = \pi^{N+1}. \tag{32}$$

In Appendix A it is shown that

$$|D(N)| = |D(N - 1)| - (1 - \pi)\pi^k|D(N - k - 1)| + \pi^N. \tag{33}$$

* $|X|$ denotes the determinant of the matrix X.

Hence,

$$\frac{|D(N)|}{\pi^{N+1}} = \frac{1}{\pi}\cdot\frac{|D(N-1)|}{\pi^{N}} - \frac{(1-\pi)\pi^{k}}{\pi^{k+1}}\cdot\frac{|D(N-k-1)|}{\pi^{N-k}} + \frac{1}{\pi}, \quad (34)$$

i.e.,

$$F_N = \frac{1}{\pi}F_{N-1} - \frac{1-\pi}{\pi}F_{N-k-1} + \frac{1}{\pi}. \quad (35)$$

The above relation is tbe desired recursive formula for tbe mean first passage time. It was obtained under the assumption that the buffer is initially empty. An alternative assumption ahout tbe initial distribution, which is also of interest, is that the buffer contains an unit symbol, i.e., $S_0 = 1$. It may be sbown that even for this case the mean first passage time satisfies the formula (35) though, of course, the initial conditions to the recursion in the formula are different.

V. PROBABILITY OF NO OVERFLOW IN A BURST

The results of this section are useful when information concerning the length of hursts is availahle. It is assumed that the distrihution of hurst length may be expressed as a weighted sum of geometric distrihutions. Given helow are formulas which yield the prohahility that the contents of the huffer during hursts do not exceed N, a given positive integer.

At this stage, assume that the distribution of hurst length is geometric; the generalization to distrihutions that are weighted sums of geometric distrihutions will he taken up later. If the burst length is denoted hy l, then

$$\Pr\{l = i\} = (1-\rho)\rho^{i-1} \quad i = 1, 2, \cdots. \quad (36)$$

for some ρ, $0 < \rho < 1$. Let $G_N \triangleq \Pr$ {buffer contents do not exceed N during a burst}. The usual decomposition into mutually exclusive events yields

$$\begin{aligned} G_N &= \sum_{m\geq 1} \Pr\{S_j \in (0, 1, \cdots, N), \quad j = 0, 1, \cdots, m; \\ &\qquad\qquad\qquad\qquad \text{and hurst lengtb} = m\} \\ &= \sum_{m\geq 1} \Pr\{X_m \cap l = m\} \\ &= \sum_{m\geq 1} \Pr\{X_m\}\Pr\{l = m\}. \end{aligned} \quad (37)$$

The last relation bolds since, $\Pr\{X_m | l = m\} = \Pr\{X_m\}$. Now,

$$\begin{aligned}\Pr\{X_m\} &= \sum_{i=0}^{N} \Pr\{S_m = i \cap X_m\} \\ &= \sum_{i=0}^{N} Q_m(i) \qquad \text{from (5),} \\ &= 1^t Q_m \\ &= 1^t A^m Q_0 \end{aligned} \tag{38}$$

where A is the $(N + 1)$ by $(N + 1)$ transition matrix defined in Section 2.1 and Q_0 is the vector given by the initial distribution–it may be assumed that $S_0 \in (0, 1, \cdots, N)$. Hence

$$\begin{aligned}G_N &= \sum_{m \geq 1} 1^t A^m Q_0 (1 - \rho)\rho^{m-1} \\ &= \frac{(1-\rho)}{\rho} 1^t \left\{ \sum_{m \geq 1} (\rho A)^m \right\} Q_0 \\ &= \frac{1-\rho}{\rho} 1^t \{(I - \rho A)^{-1} - I\} Q_0 \\ &= \frac{(1-\rho)}{\rho} \{1^t (I - \rho A)^{-1} Q_0 - 1\} \cdot \end{aligned} \tag{39}$$

In the sequel, a recursive formula for G_N is developed for tbe case where $S_0 = 0$ or, equivalently, $Q_0 = e_1$.

5.1 *Recursive Formula for G_N*

The matrix A associated witb a given N is denoted by $A(N)$. $|I - \rho A(N)| \{I - \rho A(N)\}^{-1} e_1$ is tbe vector of (signed) cofactors of the first row of $\{I - \rho A(N)\}$. Therefore, $|I - \rho A(N)| 1^t \{1 - \rho A(n)\}^{-1} e_1$ is the determinant of the matrix $B(N)$ obtained by replacing every element of the first row of $\{I - \rho A(N)\}$ by unity.

$$1^t \{1 - \rho A(N)\}^{-1} e_1 = \frac{|B(N)|}{|I - \rho A(N)|}. \tag{40}$$

Let the (signed) cofactor of tbe element $\{I - \rho A(N)\}_{1i}$ be denoted by C^{1i}, $i = 1, 2, \cdots, N + 1$. From tbe definition of $B(N)$,

$$|B(N)| = \sum_{i=1}^{N+1} C^{1i}. \tag{41}$$

The elements of the $(N+1)$th column of $\{I - \rho A(N)\}$ sum to $\{1 - \rho(1-\pi)\}$ and the elements of the remaining columns sum to $(1-\rho)$. Hence, by adding the rows $2, 3, \cdots, N+1$ to row 1, it follows that

$$
\begin{aligned}
|I - \rho A(N)| &= (1-\rho) \sum_{i=1}^{N} C^{1i} + \{1 - \rho(1-\pi)\} C^{1,N+1} \\
&= (1-\rho) \sum_{i=1}^{N+1} C^{1i} + \{1 - \rho(1-\pi) - (1-\rho)\} C^{1,N+1} \\
&= (1-\rho)|B(N)| + \rho\pi C^{1,N+1} \qquad \text{from (41)},
\end{aligned}
$$

i.e.,

$$
\frac{|B(N)|}{|I - \rho A(N)|} = \frac{1}{1-\rho} - \frac{\rho\pi}{1-\rho} \frac{C^{1,N+1}}{|I - \rho A(N)|}. \tag{42}
$$

Recapitulating,

$$
\begin{aligned}
G_N &= \frac{1-\rho}{\rho} [1^t \{1 - \rho A(N)\}^{-1} e_1 - 1] \qquad \text{from (39)} \\
&= \frac{1-\rho}{\rho} \left[\frac{|B(N)|}{|I - \rho A(N)|} - 1 \right] \qquad \text{from (40)} \\
&= 1 - \pi \frac{C^{1,N+1}}{|I - \rho A(N)|} \qquad \text{from (42)}.
\end{aligned} \tag{43}
$$

The remainder of the derivation is in two parts. First, a closed form expression for $C^{1,N+1}$ is obtained. The second part is on the recursive formula for $|I - \rho A(N)|$ and this formula is derived in Appendix B.

$$
I - \rho A(N) = \begin{array}{cc}
\begin{matrix} 1 & 2 & 3 & k+1 & k+2 & & N+1 \end{matrix} & \\
\begin{bmatrix}
(1+\lambda) & \lambda & \lambda \cdots \lambda & & & & \\
\mu & 1 & 0 & & \lambda & & \\
& \ddots & \ddots & & & \ddots & \\
& & \mu & 1 & 0 & & \lambda \\
& & & \ddots & \ddots & & \\
& & & & \mu & 1 & 0 \\
& & & & & \mu & 1
\end{bmatrix} &
\begin{matrix} 1 \\ 2 \\ \\ N-k+1 \\ \\ N \\ N+1 \end{matrix}
\end{array} \tag{44}
$$

where $-\rho(1-\pi)=\lambda$ and $-\rho\pi=\mu$. $C^{1,N+1}$, tbe (signed) cofactor to $\{I-\rho A(N)\}_{1,N+1}$, is the signed determinant of an upper triangular matrix;

$$\begin{aligned} C^{1,N+1} &= (-1)^{N+2}\mu^N = (-1)^{N+2}(-\rho\pi)^N \\ &= \rho^N\pi^N. \end{aligned} \tag{45}$$

In Appendix B it is shown tbat if x_N, a scalar, is used to denote $|I-\rho A(N)|$, then the following recursive formula holds:

$$x_N = x_{N-1} - \rho^{k+1}\pi^k(1-\pi)x_{N-k-1}. \tag{46}$$

Hence,

$$\frac{x_N}{\pi^N\rho^N} = \frac{1}{\pi\rho}\left(\frac{x_{N-1}}{\pi^{N-1}\rho^{N-1}}\right) - \frac{1-\pi}{\pi}\left(\frac{x_{N-k-1}}{\pi^{N-k-1}\rho^{N-k-1}}\right). \tag{47}$$

Let

$$y_N \triangleq \frac{x_N}{\pi^N\rho^N} = \frac{|I-\rho A(N)|}{\pi^N\rho^N} \tag{48}$$

so that,

$$y_N = \frac{1}{\pi\rho}y_{N-1} - \frac{1-\pi}{\pi}y_{N-k-1}. \tag{49}$$

From (43) and (45),

$$G_N = 1 - \frac{\pi^{N+1}\rho^N}{|I-\rho A(N)|}.$$

i.e.,

$$G_N = 1 - \frac{\pi}{y_N}. \tag{50}$$

(49) and (50) togetber provide tbe desired recursive formula for tbe probability that the contents of tbe buffer does not exceed a given level N during bursts if tbe buffer is initially empty and tbe distribution of burst lengths is geometric.

Suppose the distribution of burst lengtbs is the weigbted sum of geometric distributions; i.e.,

$$\Pr\{\text{burst length} = i\} = \sum_{j=1}^{J}\alpha_j(1-\rho_j)(\rho_j)^{i-1}. \tag{51}$$

It may then be shown that $G_N=\sum_{j=1}^J\alpha_jG_{j,N}$ wbere $G_{j,N}$ is obtained from (50) and (49) with ρ replaced by ρ_j in tbe latter equation and $G_{j,N}$ identified with G_N.

VI. MEAN TIME FOR FIRST PASSAGE IN A BURST

In Section IV certain formulas for the mean time for first passage across prespecified levels are given. In burst processes where data regarding the length of bursts is available, a more meaningful functional is one in which a level is defined to be crossed only if this event occurs during a burst. Bursts, then, may be visualized as a period of observation of the buffer. First passage across N, a positive integer, is defined to occur at i if

$$\{S_j \leqq N,\ j = 0, 1, 2, \cdots, i-1 \text{ and } S_i = N+1 \text{ and, burst length} \geqq i\}.$$

Let R_i denote the probability of this event. The functional of interest is $H_N = \sum_{i=1}^{\infty} iR_i$. The burst length distribution is assumed to be geometric; generalization to larger classes of distributions may be undertaken as indicated in the preceding section. Hence, if l denotes burst length,

$$\Pr\{l = i\} = (1-\rho)\rho^{i-1} \quad i = 1, 2, \cdots \tag{52}$$

for some ρ, $0 < \rho < 1$.

In the notation of Section 2.1,

$$\begin{aligned} R_i &= \Pr\{S_{i-1} = N \cap X_{i-1} \cap E_{i-1} = 1 \cap l \geqq i\} \\ &= \Pr\{E_{i-1} = 1\}\Pr\{S_{i-1} = N \cap X_{i-1} | l \geqq i\}\Pr\{l \geqq i\} \\ &= \pi \Pr\{S_{i-1} = N \cap X_{i-1}\}\Pr\{l \geqq i\} \\ &= \pi Q_{i-1}(N)\rho^{i-1} \\ &= \pi e_r^t(\rho A)^{i-1}Q_0. \end{aligned} \tag{53}$$

A is, of course, the $(N+1)$ by $(N+1)$ matrix defined in Section 2.1 and Q_0 is the initial condition vector.

$$H_N = \pi e_r^t\Big(\sum_{i \geqq 1} i(\rho A)^{i-1}\Big)Q_0$$

i.e.,

$$H_N = \pi e_r^t(I - \rho A)^{-1}(I - \rho A)^{-1}Q_0. \tag{54}$$

The above concludes the derivation of the closed formula for H_N–the rest of the section is concerned with recursive versions of the formula for the case where the buffer is initially empty, i.e., $Q_0 = e_1$. Once again, it is necessary to revert to the use of the symbol $A(N)$ to denote the matrix A associated with N.

For fixed N,

$$z(\rho) \triangleq e_r^t \sum_{i \geqq 0} \rho^{i+1} A^i(N) e_1$$

$$= \rho e_r^t \sum_{i \geqq 0} \{\rho A(N)\}^i e_1$$

$$= \rho e_r^t \{I - \rho A(N)\}^{-1} e_1. \tag{55}$$

Hence,

$$z'(\rho) = \frac{d}{d\rho} z(\rho) = e_r^t \sum_{i \geqq 0} (i+1) \rho^i A^i(N) e_1$$

$$= e_r^t \sum_{i \geqq 1} i \{\rho A(N)\}^{i-1} e_1$$

$$= \frac{1}{\pi} H_N. \tag{56}$$

Returning to $z(\rho)$ and (55), observe that

$$z(\rho) = \frac{\rho C^{1,N+1}}{|I - \rho A(N)|} \tag{57}$$

$$C^{1,N+1} = \rho^N \pi^N. \tag{45}$$

Hence,

$$z(\rho) = \frac{\rho^{N+1} \pi^N}{|I - \rho A(N)|} \tag{58}$$

and, from (56),

$$H_N = \frac{d}{d\rho} \left\{ \frac{\rho^{N+1} \pi^{N+1}}{|I - \rho A(N)|} \right\}.$$

Let

$$\frac{1}{v_N} \triangleq \frac{\rho^{N+1} \pi^{N+1}}{|I - \rho A(N)|}. \tag{59}$$

Since $v_N = (1/\rho\pi) y_N$ where y_N has been defined previously in (48) and the recursion in (49) for y_N is linear, v_N satisfies the same recursion. Hence, with $u_N \triangleq (d/d\rho) v_N(\rho)$, the following formula is obtained:

$$H_N = -\frac{u_N}{v_N^2} \tag{60}$$

and,

$$\begin{cases} v_N = \dfrac{1}{\rho\pi} v_{N-1} - \dfrac{1-\pi}{\pi} v_{N-k-1} & (61) \\ u_N = \dfrac{1}{\rho\pi} u_{N-1} - \dfrac{1-\pi}{\pi} u_{N-k-1} - \dfrac{1}{\rho^2\pi} v_{N-1}. & (62) \end{cases}$$

VII. ASYMPTOTICS OF RECURSIONS

The main recursions occurring in the paper are of the following forms:

$$x_N = \frac{1}{\pi} x_{N-1} - \frac{1-\pi}{\pi} x_{N-k-1} \tag{63}$$

$$y_N = \frac{1}{\pi} y_{N-1} - \frac{1-\pi}{\pi} y_{N-k-1} + \frac{1}{\pi} \tag{64}$$

$$z_N = \frac{1}{\rho\pi} z_{N-1} - \frac{1-\pi}{\pi} z_{N-k-1} \qquad \text{where } 0 < \rho < 1. \tag{65}$$

Equation (63) occurs in the formula for the (unnormalized) steady-state probabilities and in the formula for the normalization constant; (64) occurs in the formula for the mean first passage time; (65) occurs in the formula for the probability of no overflow during bursts. The fundamental solutions of these recursions are obtained from the roots of the following polynomials.

$$F(\mu) \triangleq \mu^{k+1} - \frac{1}{\pi}\mu^k + \frac{1-\pi}{\pi} \tag{66}$$

$$G(\mu) = \mu^{k+1} - \frac{1}{\rho\pi}\mu^k + \frac{1-\pi}{\pi}. \tag{67}$$

Equation (66) is associated with (63) and (64); (67) with (65). The two results given below enumerate and estimate the roots of $F(\mu)$ and $G(\mu)$ outside the unit circle.

Lemma 1[1]*: Except for one positive real root $1/\theta$, and 1, all other roots of $F(\mu)$ lie inside the unit circle $|\mu| \leqq 1$. The root $1/\theta$ lies outside the unit circle if and only if $k > \pi/(1-\pi)$.*

Lemma 1 is a specialization of a result proved in Ref. 1. Bounds on θ are also given there.

Lemma 2: $G(\mu)$ has k roots inside the unit circle $|\mu| \leqq 1$, no roots in the annular ring $1 \leqq |\mu| \leqq 1/\rho$, and one real, positive root outside the circle $|\mu| \leqq 1/\rho$.

Proof:

$$G(0) = \frac{1 - \pi}{\pi} > 0$$

$$G(1) = \frac{1}{\pi}(1 - 1/\rho) < 0$$

$$G(1/\rho) = \frac{1 - \pi}{\pi}\left[1 - \frac{1}{\rho^{k+1}}\right] < 0\,.$$

Since $G(0) > 0$ and $G(1) < 0$, there exists a real positive root of $G(\mu)$, r, where $r < 1$. Since $G(1/\rho) < 0$ and $G(\mu) \to \infty$ as $\mu \to \infty$, there exists a real positive root of $G(\mu)$, R, where $R > 1/\rho$. The following theorem which is stated without proof may now be applied.

Pellet's Theorem:[10] *Given the polynomial*

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots + a_n z^n, \quad a_p \neq 0.$$

If the polynomial

$$F_p(z) = |a_0| + |a_1|z + \cdots + |a_{p-1}|z^{p-1} \\ - |a_p|z^p + |a_{p+1}|z^{p+1} + \cdots |a_n|z^n$$

has two positive zeros r and R, $r < R$, then $f(z)$ has exactly p zeros in or on the circle $|z| < r$ and no zeros in the annular ring $r < |z| < R$.

Identifying p with k, n with $k + 1$ and $f(z)$ with $G(\mu)$ the rest of the proof follows.

The reader may now verify that, for large N,

$$\begin{aligned} x_N &\cong C_1 \qquad \text{if} \qquad k < \frac{\pi}{1-\pi} \\ &\cong C_1 + C_2 N \qquad \text{if} \qquad k = \pi/1 - \pi \\ &\cong C_1 + C_2\left(\frac{1}{\theta}\right)^N \qquad \text{if} \qquad k > \frac{\pi}{1-\pi} \end{aligned}$$

$$\begin{aligned} y_N &\cong C_1 + NC_2 \qquad \text{if} \qquad k < \frac{\pi}{1-\pi} \\ &\cong C_1 + NC_2 + N^2 C_3 \qquad \text{if} \qquad k = \pi/1 - \pi \\ &\cong C_1 + NC_2 + C_3\left(\frac{1}{\theta}\right)^N \qquad \text{if} \qquad k > \frac{\pi}{1-\pi} \end{aligned}$$

$$z_N \cong C_1(R)^N$$

where, R and $1/\theta$ are roots previously defined and the C's are constants. The constants may be obtained by fairly straightforward computations.

The qualitative difference between the forms of the expressions corresponding to $k < \pi/(1 - \pi)$ and $k > \pi/(1 - \pi)$ are noteworthy. This is not unexpected, since it may be recalled that in Ref. 1 it was proved in a more general context that the Markov chain associated with the infinite buffer is positive recurrent if and only if $k > \pi/(1-\pi)$.

VIII. A MONOTONICITY PROPERTY OF THE PROBABILITY OF A FINITE BUFFER BEING FULL

The steady-state probability of a buffer being full, i.e., $P(N)$ where N is the size of the buffer [see Section III and, in particular, eq. (21)] may be expected to be an important factor in the practical design of buffers. This is so not only because of the immediate implications of the definition but also because $1/P(N)$ is the average recurrence time of state N. However, this approach would appear to overlook the possibility that the probability of the buffer being full in the transient, i.e., in the approach to steady state, is seriously underestimated by $P(N)$. Such an event is not easy to rule out because, after all, $P(N)$ is an element of only one (normalized) eigenvector of the transition matrix while all the modes or eigenvectors and eigenvalues of the matrix contribute to yield $P_m(N)$ when m is finite. However, one of the implications of the result in this section is that, under certain conditions on the initial probability distribution of the contents of the buffer, $P(N)$ is indeed an upper bound on $P_m(N)$, i.e., $P_m(N) \leqq P(N)$, $m = 0, 1, \cdots$; furthermore, the important case of the buffer being initially empty satisfies the conditions just mentioned.

For a buffer of size N, the result states the following. Suppose at the mth instant the state probabilities satisfy the inequalities:

$$\pi P_m(i) - (1 - \pi) \sum_{j=i+1}^{i+k} P_m(j) \geqq 0 \quad i = 0, 1, \cdots, N - k \tag{68}$$

$$\pi P_m(i) - (1 - \pi) \sum_{j=i+1}^{N} P_m(j) \geqq 0$$
$$i = N - k + 1, N - k + 2, \cdots, N - 1. \tag{69}$$

Then (a) $P_m(N) \leqq P_{m+1}(N)$, and, as shown below, (b) the inequalities in (68) and (69) are satisfied with $P_m(l)$ replaced by $P_{m+1}(l)$ for $l = 0, 1, 2, \cdots, N$. Therefore, if (68) and (69) hold, $P_i(N) \leqq P_{i+1}(N)$ for all i, $i \geqq m$; i.e., the probability of the buffer being full is a monotonic,

non-decreasing sequence. (a) may be trivially verified. The proof of (b) is as follows.

(*i*) $i = 0$.

$$\begin{aligned}
&\pi P_{m+1}(i) - (1-\pi)\{P_{m+1}(i+1) + P_{m+1}(i+2) \cdots + P_{m+1}(i+k)\} \\
&\quad = \pi(1-\pi)[P_m(0) + P_m(1) + \cdots + P_m(k)] - (1-\pi)\pi \\
&\qquad \times [P_m(0) + P_m(1) + \cdots P_m(k-1)] - (1-\pi)^2[P_m(k+1) \\
&\qquad\qquad + P_m(k+2) + \cdots + P_m(2k)] \\
&\quad = (1-\pi)[\pi P_m(k) - (1-\pi)\{P_m(k+1) \\
&\qquad\qquad + P_m(k+2) \cdots + P_m(2k)\}] \geqq 0
\end{aligned}$$

(*ii*) $1 \leqq i \leqq N - 2k$.

$$\begin{aligned}
&\pi P_{m+1}(i) - (1-\pi)\{P_{m+1}(i+1) + P_{m+1}(i+2) \cdots + P_{m+1}(i+k)\} \\
&\quad = \pi(1-\pi)[P_m(0) + P_m(1) + \cdots + P_m(k)] - (1-\pi)\pi[P_m(0) \\
&\qquad + P_m(1) + \cdots + P_m(k-1)] - (1-\pi)^2[P_m(k+1) \\
&\qquad\qquad + P_m(k+2) + \cdots + P_m(2k)] \\
&\quad = (1-\pi)[\pi P_m(k) - (1-\pi)\{P_m(k+1) \\
&\qquad\qquad + P_m(k+2) + \cdots + P_m(2k)\}] \geqq 0
\end{aligned}$$

(*iii*) $N - 2k + 1 \leqq i \leqq N - k - 1$.

$$\begin{aligned}
&\pi P_{m+1}(i) - (1-\pi)\{P_{m+1}(i+1) \\
&\qquad\qquad + P_{m+1}(i+2) + \cdots + P_{m+1}(i+k)\} \\
&\quad = \pi[P_m(i-1) + (1-\pi)P_m(i+k)] - (1-\pi) \\
&\qquad \times \pi[P_m(i) + P_m(i+1) + \cdots + P_m(i+k-1)] \\
&\qquad - (1-\pi)^2[P_m(i+k+1) + P_m(i+k+2) + \cdots + P_m(N)] \\
&\quad = \pi[\pi P_m(i-1) - (1-\pi)\{P_m(i) + P_m(i+1) \\
&\qquad + \cdots + P_m(i+k-1)\}] + (1-\pi)[\pi P_m(i+k) - (1-\pi) \\
&\qquad\quad \times \{P_m(i+k+1) + P_m(i+k+2) + \cdots + P_m(N)\}] \geqq 0
\end{aligned}$$

(*iv*) $i = N - k$.

$$\begin{aligned}
&\pi P_{m+1}(i) - (1-\pi)\{P_{m+1}(i+1) \\
&\qquad\qquad + P_{m+1}(i+2) + \cdots + P_{m+1}(i+k)\} \\
&\quad = \pi[\pi P_m(i-1) + (1-\pi)P_m(i+k)] - (1-\pi)\pi[P_m(i) \\
&\qquad\quad + P_m(i+1) + \cdots + P_m(i+k-1)] - (1-\pi)\pi P_m(N) \\
&\quad = \pi[\pi P_m(i-1) - (1-\pi)\{P_m(i) \\
&\qquad\qquad + P_m(i+1) + \cdots + P_m(i+k-1)\}] \geqq 0
\end{aligned}$$

(v) $N - k + 1 \leqq i \leqq N - 1$.

$$
\begin{aligned}
\pi P_{m+1}(i) - (1-\pi)\{P_{m+1}(i+1) + P_{m+1}(i+2) + \cdots + P_{m+1}(N)\} \\
= \pi^2 P_m(i-1) - (1-\pi)\pi[P_m(i) + P_m(i+1) + \cdots + P_m(N)] \\
= \pi[\pi P_m(i-1) - (1-\pi)\{P_m(i) + P_m(i+1) \\
+ \cdots + P_m(N)\}] \geqq 0
\end{aligned}
$$

(b) is proved.

Observe that $P_m(0) = 1$, $P_m(i) = 0$, $i = 1, 2, \cdots, N$ satisfies the inequalities (68) and (69). However, the other initial distribution of interest, namely, $P_m(0) = 0$, $P_m(1) = 1$, $P_m(i) = 0$, $i = 2, 3, \cdots, N$ does not satisfy the inequalities. Also, it may be verified that for the latter set of initial conditions, the monotonicity property does not hold.

It is interesting to note that if (68) and (69) hold, then $P_m(0) \geqq P_{m+1}(0)$, so that together with (b), $P_i(0) \geqq P_{i+1}(0)$ for all i, $i \geqq m$, i.e., the probability of the buffer being empty is a monotonic, non-increasing sequence.

APPENDIX A

Recursive Formula for $|D(N)|$

$$
D(N) = \begin{array}{c}
\begin{array}{cccccccccc}
1 & 2 & & & k+1 & & & & N & N+1
\end{array} \\
\left[\begin{array}{cccccccccc}
1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 \\
-\pi & 1 & 0 & & (\pi-1) & 0 & & & & \\
 & \ddots & \ddots & & & \ddots & \ddots & & & \\
 & & & -\pi & 1 & 0 & & & & (\pi-1) \\
 & & & & -\pi & 1 & 0 & & & 0 \\
 & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & & & 1 & 0 \\
 & & & & & & & & -\pi & 1
\end{array}\right]
\begin{array}{l}
1 \\ 2 \\ \\ N-k+1 \\ N-k+2 \\ \\ N \\ N+1
\end{array}
\end{array}
$$

Expanding $|D(N)|$ along the $(N+1)$st row yields

$$
|D(N)| = |D(N-1)| + \pi|X| \tag{70}
$$

where

$$
X \triangleq
\begin{array}{cc}
\begin{array}{ccccccccc} 1 & 2 & & k+1 & & & & N-1 & N \end{array} & \\
\left[\begin{array}{ccccccccc}
1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1\\
-\pi & 1 & 0 & (\pi-1) & & & & & \\
 & \ddots & \ddots & & \ddots & & & & \\
 & & -\pi & 1 & 0 & & \cdots & (\pi-1) & 0\\
 & & & -\pi & 1 & 0 & & 0 & 0\\
 & & & & -\pi & 1 & 0 & 0 & (\pi-1)\\
 & & & & & \ddots & \ddots & & \\
 & & & & & & -\pi & 1 & 0\\
 & & & & & & & -\pi & 0
\end{array}\right]
& \begin{array}{c} 1\\ 2\\ \\ N-k-1\\ N-k\\ N-k+1\\ \\ \\ N\end{array}
\end{array}
$$

Expanding $|X|$ along last column:

$$|X| = (-1)^{N+1}(-\pi)^{N-1} + (\pi - 1)(-1)^{N+(N-k+1)}|Y| \tag{71}$$

where

$$
Y \triangleq
\begin{array}{cc}
\begin{array}{cccccccc} 1 & 2 & & k+1 & & & & N-1 \end{array} & \\
\left[\begin{array}{cccccccc}
1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1\\
-\pi & 1 & 0 & \pi-1 & & & & \\
 & \ddots & \ddots & & \ddots & & & \\
 & & -\pi & 1 & 0 & & \cdots & \pi-1\\
 & & & -\pi & 1 & & & 0\\
 & & & & \ddots & \ddots & & \\
 & & & & & & -\pi & 1\\
 & & & & & & & -\pi
\end{array}\right]
& \begin{array}{c} 1\\ 2\\ \\ N-k-1\\ N-k\\ \\ \\ N-1\end{array}
\end{array}
$$

Expanding $|Y|$ along the last $(k - 1)$ row.

$$|Y| = (-\pi)^{k-1}|D(N - k - 1)|. \tag{72}$$

Combining (70) through (72):

$$|D(N)| = |D(N-1)| - (1-\pi)\pi^k |D(N-k-1)|. \tag{73}$$

APPENDIX B

Recursive Formula for $|I - \rho A(N)|$

$I - \rho A(N)$ is given in (44). Let x_n denote $|I - \rho A(N)|$. Also,

$$\lambda = -\rho(1-\pi) \quad \text{and} \quad \mu = -\rho\pi.$$

(*i*) Expand $|I - \rho A(N)|$ along the last, i.e., $(N+1)$th row, of $[I - \rho A(N)]$.

$$x_N = x_{N-1} - \mu |X| \tag{74}$$

where

$$X \triangleq \begin{array}{cl}
\begin{array}{cccccccc} 1 & 2 & \quad & k+1 & \quad & \quad & \quad & N \end{array} & \\
\left[\begin{array}{cccccccc}
1+\lambda & \lambda & \cdots & \lambda & 0 & & & 0 \\
\mu & 1 & 0 & & \lambda & & & \\
 & \ddots & \ddots & & & \ddots & & \\
 & & & & & & \lambda & 0 \\
 & & & & & & 0 & 0 \\
 & & & & & & 0 & \lambda \\
 & & & \ddots & & \ddots & & \\
 & & & & & \mu & 1 & 0 \\
 & & & & & & \mu & 0
\end{array}\right] &
\begin{array}{l} 1 \\ 2 \\ \\ N-k-1 \\ N-k \\ N-k+1 \\ \\ \\ N \end{array}
\end{array}$$

(*ii*) Expand $|X|$ along the last, i.e., Nth column of X.

$$\begin{aligned} |X| &= (-1)^{(N-k+1)+N}\lambda |Y| \\ &= (-1)^{k-1}\lambda |Y| \end{aligned} \tag{75}$$

where

$$Y \triangleq \begin{array}{cl}
\begin{array}{ccccccccc} 1 & 2 & \quad & k+1 & \quad & \quad & \quad & \quad & \quad \end{array} & \\
\left[\begin{array}{ccccccccc}
1+\lambda & \lambda & \cdots & \lambda & & & & & \\
\mu & 1 & & & \lambda & & & & \\
 & \ddots & \ddots & & & \ddots & & & \\
 & & & \mu & 1 & 0 & & & \lambda \\
 & & & & \mu & 1 & 0 & & 0 \\
 & & & & 0 & \mu & 1 & 0 & 0 \\
 & & & & & & \ddots & \ddots & \\
 & & & & & & & \mu & 1 \\
 & & & & & & & & \mu
\end{array}\right] &
\begin{array}{l} 1 \\ 2 \\ \\ N-k-1 \\ N-k \\ N-k+1 \\ \\ \\ N-1 \end{array}
\end{array}$$

(*iii*) Expand $|Y|$ along the last $k - 1$ rows.

$$|Y| = \mu^{k-1}x_{N-k-1}. \tag{76}$$

Hence,

$$\begin{aligned} x_N &= x_{N-1} - \mu|X| && \text{from (74)} \\ &= x_{N-1} - \mu(-1)^{k-1}\lambda|Y| && \text{from (75)} \\ &= x_{N-1} - \mu(-1)^{k-1}\lambda\mu^{k-1}x_{N-k-1} && \text{from (76)} \\ &= x_{N-1} - \rho^{k+1}\pi^k(1 - \pi)x_{N-k-1}. && (77) \end{aligned}$$

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